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Attractor Behavior of Phantom Cosmology

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Abstract

We investigate the cosmological attractor of the minimally coupled, self-interacting phantom field with a positive energy density but negative pressure. It is proved that the phantom cosmology is rigid in the sense that there exists a unique attractor solution. We plot the trajectories in the phase space numerically for the phantom field with three typical potentials. Phase portraits indicate that an initial kinetic term decays rapidly and the trajectories reach the unique attractor curve. We find that the curve corresponds to the slow-climb solution.

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A scalar field with negative kinetic energy is proposed to explain the accelerated expansion of present universe as dark energy [1]. This form of dark energy with the state equation parameter $w < -1$, dubbed phantom energy violates the dominant-energy condition. It was shown that this model is consistent with both recent observations and classical tests of cosmology, in some cases providing a better fit than the more familiar models with $w > -1$. In spite of the fact that the field theory of phantom fields encounters the problem of stability which one could try to bypass by assuming them to be effective fields [2, 3], it is nevertheless interesting to study their cosmological implication. Recently, there are many relevant studies of phantom energy [4] and the primordial perturbation spectrum from various phantom inflation models [5].

The physical background for phantom type of matter with strongly negative pressure may be looked for in string theory [6]. Phantom field may also arise from a bulk viscous stress due to particle production [7] or in higher-order theories of gravity [8], Brans-Dicke and non-minimally coupled scalar field theories [9]. The cosmological models which allow for phantom matter appear naturally in the mirage cosmology of the braneworld scenario [10] and in k-essence models [11].

In this letter we study the attractor behavior of phantom cosmology. Using the Hamilton-Jacobi formalism, we prove that there exists a unique attractor solution in the early universe containing a minimally coupled, self-interacting phantom field. We use an explicit numerical computation of the phase space trajectories for the phantom field with three typical potentials. Phase portraits indicate that the initial kinetic term decays rapidly and the trajectories reach the unique attractor curve. The attractor curve corresponds to the slow-climb solution.

The action of the phantom field minimally coupled to gravity can be written as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right), \tag{1}$$

where $\kappa^2 \equiv 8\pi G_N$ is the gravitational coupling, $V(\phi)$ is the phantom potential and the metric signature $(-,+,+,+)$ is employed. The climbing phantom in a spatially flat FRW cosmological model can be described by a fluid with a positive energy density ρ and a negative pressure P given by

$$\rho = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2}$$

$$P = -\frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{3}$$

which means that phantom energy violates the dominant energy condition, $\rho + P < 0$ and $\rho > 0$. The corresponding equation of state parameter is now given by

$$w \equiv \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)}. \tag{4}$$

Since the phantom energy density is positive, Eq.(4) indicates that $w < -1$, and $w \to -1$ as the ratio $\dot{\phi}^2/2V(\phi) \to 0$. The evolution equation of the phantom field and the Friedmann constraint are

$$\ddot{\phi} + 3H\dot{\phi} - V'(\phi) = 0, \tag{5}$$

$$H^2 = \frac{\kappa^2}{3}\left[-\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \tag{6}$$

Note that the sign of the potential force term is negative in Eq.(5), which distinguishes the phantom field from the ordinary field and implies that the phantom field climbs up the potential.

The Hamilton-Jacobi formulation is a powerful way of rewriting the equations of motion, which allows an easier derivation of many inflation results. We concentrate here on the homogeneous situation as applied to spatially flat cosmologies, and demonstrate the stability of the phantom cosmology using the Hamilton-Jacobi formalism [12].

Differentiating Eq.(6) with respect to t and substituting in Eq.(5) gives

$$\dot{\phi} = \frac{2}{\kappa^2}H'(\phi). \tag{7}$$

This allows us to write the Friedmann equation in the first-order form

$$[H'(\phi)]^2 + \frac{3}{2}\kappa^2 H^2(\phi) = \frac{\kappa^4}{2}V(\phi). \tag{8}$$

Eqs.(7) and (8) are the Hamilton-Jacobi equations. They allow us to consider $H(\phi)$, rather than $V(\phi)$, as the fundamental quantity to be specified.

Suppose $H(\phi, p)$ is the solution of Eq.(8) which is uniquely determined once the initial conditions have been specified, where the parameter p is associated with each solution. The general solution to Eq.(7) can be expressed as

$$a(\phi, p) = a_i \exp\left[\frac{\kappa^2}{2}\int_{\phi_i}^{\phi} d\phi H(\phi, p)\left(\frac{\partial H(\phi, p)}{\partial \phi}\right)^{-1}\right], \tag{9}$$

where a_i is the value at some initial point ϕ_i. We consider two solutions $H(\phi, p + \Delta p)$ and $H(\phi, p)$, where $\Delta p \ll 1$. By differentiating Eq.(8) with respect to p and using Eq.(9), we obtain

$$H(\phi, p + \Delta p) - H(\phi, p) \propto a^{-3}(\phi, p)\Delta p. \tag{10}$$

We find that any two solutions approach the attractor solution $H(\phi)$ in an expanding universe. However, the attractor may not be the same for all values of the parameters. In order to check whether there exists a unique attractor, one defines the quantity [13]

$$F \equiv \left|\frac{H(\phi, p + \Delta p)}{H(\phi, p)} - 1\right|. \tag{11}$$

Eq.(10) shows that $F \to 0$ as the universe expands, but the form of the attractor may vary if $\partial F/\partial\phi$ changes sign. Note that F can go to zero for $\partial F/\partial\phi > 0$ or for $\partial F/\partial\phi < 0$, but not for both. Thus, if $\partial F/\partial\phi = 0$ for some value of the parameters, the attractor solution will not be unique for all values of the parameters.

$$\begin{aligned} \frac{\partial \ln F}{\partial \phi} &= -H(\phi,p)\left(\frac{\partial H(\phi,p)}{\partial\phi}\right)^{-1} \\ &\quad \times \left[\frac{3\kappa^2}{2} + \frac{1}{H^2(\phi,p)}\left(\frac{\partial H(\phi,p)}{\partial\phi}\right)^2\right], \end{aligned} \tag{12}$$

which indicates that there exists a unique attractor because $\partial F/\partial\phi$ can not pass through zero.

To study an explicit numerical computation of the phase space trajectories, it is most convenient to rewrite the evolution Eqs.(5) and (6) for H and ϕ as a set of two first-order differential equations with two independent variables ϕ and $\dot\phi$

$$\frac{d\phi}{dt} = \dot\phi, \tag{13}$$

$$\frac{d\dot\phi}{dt} = -\sqrt{3}\kappa\left[-\frac{1}{2}\dot\phi^2 + V(\phi)\right]^{\frac{1}{2}}\dot\phi + V'(\phi). \tag{14}$$

Let us consider three typical potentials for the phantom field: a quadratic potential $V(\phi) = m^2\phi^2/2$, an exponential potential $V(\phi) = V_0\exp(-\lambda\kappa\phi)$ and a potential of the form $V(\phi) = V_0[1+\cos(\phi/f)]$. We choose different initial conditions ϕ_0 and $\dot\phi_0$ in the range $\dot\phi_0^{\,2} < 2V(\phi_0)$, and obtain the phase portraits (Fig.1, Fig.2 and Fig.3, respectively) in the $(\phi,\dot\phi)$ phase plane.

Power-law potential. In Figure 1, we see that there is a unique curve that attracts most of the trajectories in each branch of the two non-connected regions. An initial kinetic term decays rapidly. The phantom field climbs up the potential, which differs from the rolling-down behavior of the normal scalar field. The two attractor curves correspond to the slow-climb solutions:

$$\dot\phi = \frac{\sqrt{6}m}{3\kappa} \tag{15}$$

in the region $\phi > 0$ and $-m\phi < \dot\phi < m\phi$, and

$$\dot\phi = -\frac{\sqrt{6}m}{3\kappa} \tag{16}$$

in the region $\phi < 0$ and $m\phi < \dot\phi < -m\phi$. In the model, the ratio of kinetic to potential energy of the phantom field tends to zero, so $w \to -1$. The general features of the behavior of the trajectories do not change when we use a power-law potential $V(\phi) = \lambda\phi^\alpha$. Although

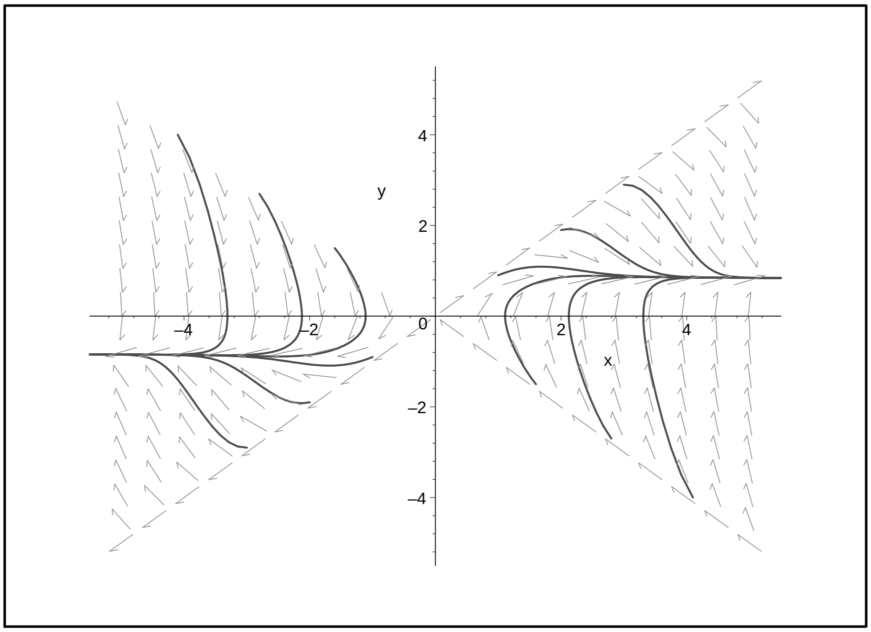

Figure 1: Phase portrait for the phantom field with the potential $V(\phi) = m^2\phi^2/2$ in the $(\phi, \dot{\phi})$ phase plane.

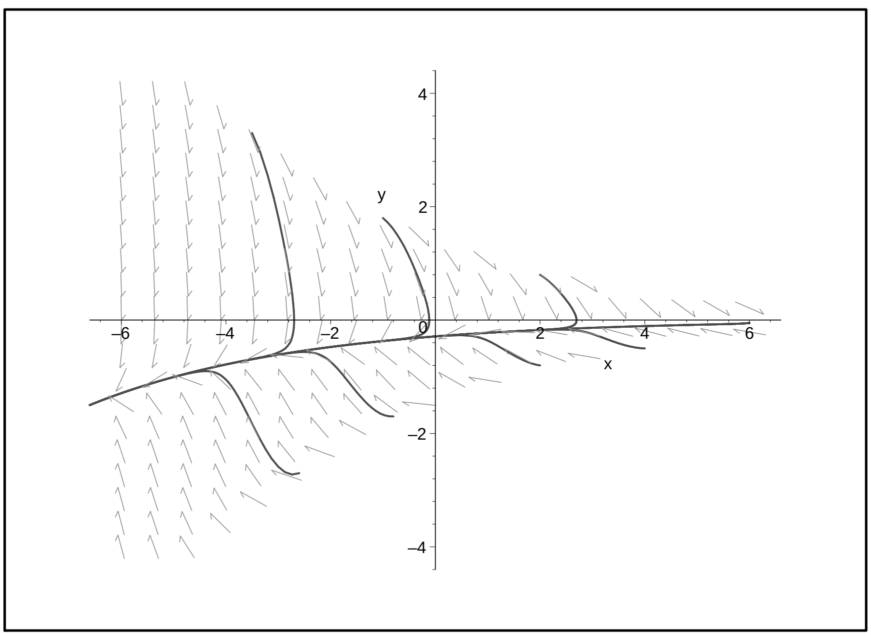

Figure 2: Phase portrait for the phantom field with the potential $V(\phi) = V_0 \exp(-\lambda\kappa\phi)$ in the $(\phi, \dot{\phi})$ phase plane.

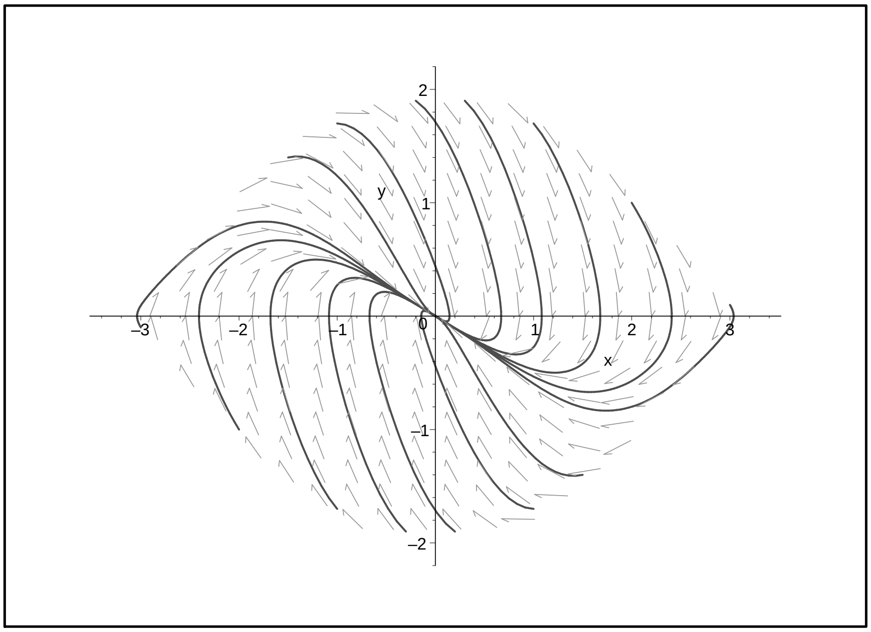

Figure 3: Phase portrait for the phantom field with the potential $V(\phi) = V_0[1 + \cos(\phi/f)]$ in the $(\phi, \dot{\phi})$ phase plane.

for $\alpha > 2$ the kinetic term is no more a constant value and increases as the universe expands, the ratio of kinetic to potential energy still tends to zero proportional to ϕ^{-2}. For $\alpha = 4$ the slow-climb regime becomes the regime of an exponential growth of the phantom field.

Exponent potential. In Figure 2, the attractor curve corresponds to the slow-climb solution:

$$\dot{\phi} = -\lambda\sqrt{\frac{V_0}{3}}e^{-\frac{1}{2}\lambda\kappa\phi} \tag{17}$$

in the region $-\sqrt{2V_0}\exp(-\lambda\kappa\phi/2) < \dot{\phi} < \sqrt{2V_0}\exp(-\lambda\kappa\phi/2)$. In the model, though the kinetic term grows exponentially the ratio of kinetic to potential energy of the phantom field is a constant value $\lambda^2/6$, so $w = (\lambda^2 + 6)/(\lambda^2 - 6) < -1$. Therefore, for a phantom field with an exponential potential $V(\phi) = V_0\exp(-\lambda\kappa\phi)$, the scaling solution exists and is stable as long as $\lambda^2 < 6$, while for $\lambda^2 > 6$ the slow-climb approximation breaks down. This regime is analogous to the scaling solution for a normal scalar field with an exponential potential [14].

Cosine potential. In Figure 3, the attractor curve corresponds to the slow-climb solution:

$$\dot{\phi} = -\frac{\sqrt{V_0}}{\sqrt{3}\kappa f}\left(1 + \cos\frac{\phi}{f}\right)^{-\frac{1}{2}}\sin\frac{\phi}{f} \tag{18}$$

in the region $-\sqrt{2V_0}[1 + \cos(\phi/f)]^{1/2} < \dot{\phi} < \sqrt{2V_0}[1 + \cos(\phi/f)]^{1/2}$. The phantom field

moves towards the top of the potential and then settles at $\phi = 0$. The friction term $3H\dot{\phi}$ is so strong compared to the force term $V'(\phi)$ that the phantom field can not oscillate about the maximum of the potential, which differs from the regime for a gaussian potential [2]. In this case, the ratio of kinetic to potential energy of the phantom field tends to zero, so the state equation parameter w tends to -1. Thus the de-Sitter like solution is the late-time attractor of the model.

In conclusion, we have investigated the attractor behavior of phantom cosmology. It is proved that there exists a unique attractor solution, hence the dynamical system of a single, minimally coupled phantom field is rigid irrespective of the nature of the potential. Moreover, we consider three different model: power-law potential, exponential potential and cosine potential, and plot the trajectories in the phase space numerically. Phase portraits indicate that an initial kinetic term decays rapidly and the trajectories reach the unique attractor curve. We find that the curve corresponds to the slow-climb solution. The work can be extended to the braneworld scenario.

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